Rideal cannot have been right in suggesting that all reported values of α less than unity are the result of errors in measuring surface temperature. α appears to be a property of the evaporating substance, decreasing in value with increasing temperature.

NOTATION

A = constant in Equation (2)
A_s = area of surface of test piece, sq. cm.
A_w = area of condenser surface, sq. cm.
B = constant in Equation (2)
$\dot{m}$ = evaporation flux, g./(sec.)(sq. cm.)
M = molecular weight of evaporating substance
P = vapor pressure, mm. Hg.
P_s = vapor pressure at surface temperature, mm. Hg.
R = gas constant
T = temperature, °K.
T_b = temperature of blackbody cavity, °K.
T_s = temperature of surface, °K.
T_w = temperature of surrounding condenser, °K.
α = evaporation coefficient, defined by Equation (1)
ϵ = emissivity of coated test specimen
σ = Stefan-Boltzmann constant
λ = enthalpy of sublimation, cal./g.

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Transfer Functions of Heat Exchangers

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In the course of research the authors have collected a number of transfer functions for various types of heat exchangers. For the convenience of analysis the heat exchangers may be classified into several basic types described in subsequent pages. In each case certain assumptions are made when one writes down the differential equations of the heat exchanger, which constitute the so-called *mathematical model* of the exchanger. If a particular heat exchanger is being classified, it is important to know the nature of temperature variation and the relative directions of flow of the exchanging fluid streams. If the heat exchanger in question matches one of the types described, then the transfer functions listed may be used to obtain its theoretical frequency and transient responses.

To obtain the frequency response the Laplace transform variable s in the transfer function, $G(s)$ is replaced by $j\omega$, and the moduli and arguments of $G(j\omega)$ for different values of ω are, respectively, the amplitude ratios of the output to the input and the phase angles between them. To obtain the transient response for a unit step input it is necessary to invert the function $G(s)/s$, and this procedure is not always easy. Most of the transfer functions presented here appeared previously in the literature, but some were obtained by the present authors. This paper will present to the research workers in this field a convenient collection of available transfer functions, thus eliminating the necessity of duplicating the burdensome mathematical work.

The following notes apply to the notation and procedures used in all cases. The independent time variable t and space variable x are nondimensionalized to τ and ξ, and the system parameters are grouped in dimensionless forms denoted by a's and b's. Certain uniformity and symmetry of notations are incorporated in this presentation so that the reader, in passing from one type of heat exchanger to another, will have little difficulty in interpreting the notations encountered. The temperature variables θ's and ϕ's in all the differential equations denote the time varying part of the temperatures, that is the part representing the deviations or perturbations of the temperatures from their steady state values; therefore when one assumes that the system is in the steady state at the time $t = 0$, the initial values of all these variables are zero.

To derive the transfer functions of the heat exchanger the inlet temperature of one fluid stream is regarded as the input or forcing function of the exchanger and the outlet temperature of the same or other fluid stream as the output function of the exchanger. The transfer function is defined as the ratio of the Laplace transform of the output function to that of the input function, provided that the system is initially in the steady state. The same transfer function may be regarded as the Laplace transform of the output function when the input function assumes a unit impulse function $\delta(t)$. This definition is used to formulate the boundary conditions for the heat exchanger to obtain the desired transfer functions. The differential equations are linearized and

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then solved in the s-domain, a procedure which can be justified for small perturbations.

Where the temperatures of the fluid and the tube wall vary along the length of the heat exchanger, the following assumptions are made: (a) the velocity and temperature of the fluid are uniform across the cross-sectional area of the tube, (b) the heat transfer within the fluid in the longitudinal direction of flow due to molecular conduction and turbulence can be neglected, (c) the mechanical energy (kinetic and potential) of the fluid is negligible compared with the thermal energy, (d) the fluid is essentially incompressible, (e) the temperature of the tube (or shell) walls is uniform throughout the cross section normal to the direction of flow of the fluid (reasonable for thin metallic walls), (f) the heat conduction in the tube wall in the longitudinal direction can be neglected. It is further assumed that the outside surface of the shell wall is perfectly insulated. Most differential equations are obtained by applying energy balance on the exchanging streams with the above assumptions.

TYPE I. BOTH FLUIDS THOROUGHLY MIXED OR ISOTHERMAL

This type of heat exchanger, schematically represented in Figure 1, has been analyzed by Mozley (*8*), Takahashi (*9*), and Campbell (*1*). Very few heat exchangers in practice belong to this type; however where the fluid temperature changes continuously along the length of the heat exchanger, the exchanger may be divided into several sections within which the temperature of the fluid may be considered uniform, with the derived equations applying to the individual sections. The technique of studying the heat exchanger with continuously varying temperatures (distributed systems) by lumping the system parameters into several sections has been employed by Mozley (*8*) and forms the basis for analogue computation.

The differential equations describing this system are

$$\frac{d\theta_1}{d\tau} + \theta_1 - \theta_{1i} = a_1 (\phi - \theta_1)$$
$$\frac{d\phi}{d\tau} = b_1 (\theta_1 - \phi) + b_2 (\theta_2 - \phi) \quad (1)$$
$$r\frac{d\theta_2}{d\tau} + \theta_2 - \theta_{2i} = a_2 (\phi - \theta_2)$$

where

$$a_1 = \frac{h_1 A_1}{w_1 c_1}, \qquad a_2 = \frac{h_2 A_2}{w_2 c_2}$$
$$b_1 = \frac{h_1 A_1 M_1}{w_1 c_w M_w}, \qquad b_2 = \frac{h_2 A_2 M_1}{w_1 c_w M_w} \quad (2)$$

$$\tau = t/T_1,\ T_1 = M_1/w_1,\ T_2 = M_2/w_2$$
$$r = T_2/T_1 = M_2 w_1/M_1 w_2$$

The boundary conditions are

Case (a). Inlet temperature of stream 2 steady and inlet temperature of stream 1 as forcing function:

$$\theta_1 (0) = \theta_2 (0) = 0$$
$$\theta_{1i} (\tau) = \delta (\tau) \quad (3)$$
$$\theta_{2i} (\tau) = 0$$

Case (b). Inlet temperature of stream 1 steady and inlet temperature of stream 2 as forcing function:

$$\theta_1 (0) = \theta_2 (0) = 0$$
$$\theta_{1i} (\tau) = 0 \quad (4)$$
$$\theta_{2i} (\tau) = \delta (\tau)$$

The corresponding transfer functions are

Case (a):

$$\frac{\bar{\theta}_1(s)}{\bar{\theta}_{1i}(s)} = \frac{f_2}{f_1 f_2 - g_1 g_2} \quad (5)$$
$$\frac{\bar{\theta}_2(s)}{\bar{\theta}_{1i}(s)} = \frac{g_2}{f_1 f_2 - g_1 g_2} \quad (6)$$

Case (b):

$$\frac{\bar{\theta}_1(s)}{\bar{\theta}_{2i}(s)} = \frac{g_1}{f_1 f_2 - g_1 g_2} \quad (7)$$
$$\frac{\bar{\theta}_2(s)}{\bar{\theta}_{2i}(s)} = \frac{f_1}{f_1 f_2 - g_1 g_2} \quad (8)$$

where

$$f_1 = 1 + a_1 - \frac{a_1 b_1}{b_1 + b_2 + s} + s,$$
$$f_2 = 1 + a_2 - \frac{a_2 b_2}{b_1 + b_2 + s} + rs$$
$$g_1 = \frac{a_1 b_2}{b_1 + b_2 + s}, \qquad g_2 = \frac{a_2 b_1}{b_1 + b_2 + s} \quad (9)$$

If the thermal capacity of the partition wall can be neglected, the temperature of the wall does not enter into the picture. Therefore the second equation of (1) disappears, and the system equations reduce to

$$\frac{d\theta_1}{d\tau} + \theta_1 - \theta_{1i} = a_1' (\theta_2 - \theta_1)$$
$$(10)$$
$$r\frac{d\theta_2}{d\tau} + \theta_2 - \theta_{2i} = a_2' (\theta_1 - \theta_2)$$

where

$$a_1' = U_2 A_2/w_1 c_1,\ a_2' = U_2 A_2/w_2 c_2 \quad (11)$$
$$U_2 = \frac{1}{A_2/h_1 A_1 + 1/h_2 + yA_2/kA_m}$$

Here the overall coefficient U is arbitrarily defined on the basis of the surface A_2 in contact with stream 2. Although the wall heat capacity is neglected its thermal resistance may be included in computing U, as given by (11).

The transfer functions for this case are the same as given by (5) to (8) with the following modifications for f_1, f_2, g_1, and g_2:

$$f_1 = 1 + a_1' + s, \qquad f_2 = 1 + a_2' + rs$$
$$g_1 = a_1' \qquad g_2 = a_2' \quad (12)$$

TYPE 2. 1-1 HEAT EXCHANGER, TEMPERATURES OF BOTH STREAMS CONTINUOUSLY VARYING ALONG THE PATH (NO PHASE CHANGE OF EITHER STREAM)

This type of heat exchanger consists of one tube path and one shell path and is commonly known as *double-pipe heat exchanger*. It may also refer however, to an exchanger with several parallel tube passes inside a shell. Two orientations of the relative directions of flow of the tube and the shell streams are possible, namely the parallel flow and the counterflow, as depicted in Figures 2*a* and 2*b*, respectively.

Some of the eight transfer functions introduced below were given by Gould (*5*) and Takahashi (*9*), who also studied this type of heat exchanger experimentally.

The differential equations for this type of heat exchanger are

$$\frac{\partial\theta_1}{\partial\tau} \pm \frac{\partial\theta_1}{\partial\xi} = a_1 (\phi - \theta_1)$$
$$r\frac{\partial\theta_2}{\partial\tau} + \frac{\partial\theta_2}{\partial\xi} = a_2 (\phi - \theta_2) + a_s (\phi_s - \theta_2)$$
$$\frac{\partial\phi}{\partial\tau} = b_1 (\theta_1 - \phi) + b_2 (\theta_2 - \phi)$$
$$\frac{\partial\phi_s}{\partial\tau} = b_s (\theta_2 - \phi_s) \quad (13)$$

where

$$a_1 = \frac{h_1 A_1 L}{u_1 S_1 \rho_1 c_1}, a_2 = \frac{h_2 A_2 L}{u_2 S_2 \rho_2 c_2}, a_s = \frac{h_s A_s L}{u_2 S_2 \rho_2 c_2},$$
$$b_1 = \frac{h_1 A_1 L}{u_1 c_t}, b_2 = \frac{h_2 A_2 L}{u_1 c_t}, b_s = \frac{h_s A_s L}{u_1 c_s},$$
$$\xi = x/L, \qquad \tau = tu_1/L \qquad r = u_1/u_2 \quad (14)$$

The boundary conditions are

Case (a). Parallel flow, inlet temperature of shell fluid steady, inlet temperature of tube fluid as forcing function:

$$\theta_1 (\xi, \tau = 0) = \theta_2 (\xi, \tau = 0) = 0$$
$$\theta_1 (0, \tau) = \delta (\tau) \quad (15)$$
$$\theta_2 (0, \tau) = 0$$

Case (b). Parallel flow, inlet temperature of tube fluid steady, inlet temperature of shell fluid as forcing function:

$$\theta_1(\xi, \tau = 0) = \theta_2(\xi, \tau = 0) = 0$$
$$\theta_1(0, \tau) = 0 \qquad (16)$$
$$\theta_2(0, \tau) = \delta(\tau)$$

Case (c). Counterflow, inlet temperature of shell fluid steady, inlet temperature of tube fluid as forcing function:

$$\theta_1(\xi, \tau = 0) = \theta_2(\xi, \tau = 0) = 0$$
$$\theta_1(1, \tau) = \delta(\tau)$$
$$\theta_2(0, \tau) = 0 \qquad (17)$$

Case (d). Counterflow, inlet temperature of tube fluid steady, inlet temperature of shell fluid as forcing function:

$$\theta_1(\xi, \tau = 0) = \theta(\xi, \tau = 0) = 0$$
$$\theta_1(1, \tau) = 0 \qquad (18)$$
$$\theta_2(0, \tau) = \delta(\tau)$$

In Equation (13) the direction of flow of the shell fluid has been chosen as the positive direction of the x axis; hence the + sign in the first equation of (13) applies to the parallel flow case and the − sign the counterflow case.

If one solves (13) together with the boundary conditions for the different cases, the following transfer functions are obtained.

Parallel Flow

Cases (a) and (b):

$$\frac{\bar{\theta}_{2o}(s)}{\bar{\theta}_{1i}(s)} = \frac{g_2}{\lambda_1 - \lambda_2}\,(e^{\lambda_1} - e^{\lambda_2}) \qquad (19)$$

$$\frac{\bar{\theta}_{1o}(s)}{\bar{\theta}_{1i}(s)} = \frac{g_1 g_2}{\lambda_1 - \lambda_2}\left(\frac{1}{f_1 + \lambda_1} e^{\lambda_1} - \frac{1}{f_1 + \lambda_2} e^{\lambda_2}\right) \qquad (20)$$

$$\frac{\bar{\theta}_{1o}(s)}{\bar{\theta}_{2i}(s)} = \frac{g_1}{\lambda_1 - \lambda_2}\,(e^{\lambda_1} - e^{\lambda_2}) \qquad (21)$$

$$\frac{\bar{\theta}_{2o}(s)}{\bar{\theta}_{2i}(s)} = -\frac{g_1 g_2}{\lambda_1 - \lambda_2}\left(\frac{1}{f_1 + \lambda_2} e^{\lambda_1} - \frac{1}{f_1 + \lambda_1} e^{\lambda_2}\right) \qquad (22)$$

where

$$\lambda_1, \lambda_2 = -(f_1 + f_2)/2 \pm \sqrt{(f_1 - f_2)^2 + 4 g_1 g_2}/2 \qquad (23)$$

$$f_1 = a_1 (b_2 + s)/(b_1 + b_2 + s) + s$$
$$f_2 = a_2 (b_1 + s)/(b_1 + b_2 + s) + a_s\, s/(b_s + s) + rs$$
$$g_1 = a_1 b_2/(b_1 + b_2 + s)$$
$$g_2 = a_2 b_1/(b_1 + b_2 + s) \qquad (24)$$

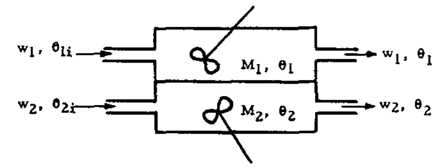

Fig. 1. Type I heat exchanger, both fluids thoroughly mixed.

Counterflow Case

Cases (c) and (d):

$$\frac{\bar{\theta}_{1o}(s)}{\bar{\theta}_{1i}(s)} = \frac{\lambda_1 - \lambda_2}{f_1(e^{\lambda_1} - e^{\lambda_2}) - \lambda_2 e^{\lambda_1} + \lambda_1 e^{\lambda_2}} \qquad (25)$$

$$\frac{\bar{\theta}_{2o}(s)}{\bar{\theta}_{1i}(s)} = \frac{g_2(e^{\lambda_1} - e^{\lambda_2})}{f_1(e^{\lambda_1} - e^{\lambda_2}) - \lambda_2 e^{\lambda_1} + \lambda_1 e^{\lambda_2}} \qquad (26)$$

$$\frac{\bar{\theta}_{1o}(s)}{\bar{\theta}_{2i}(s)} = \frac{g_1(e^{\lambda_1} - e^{\lambda_2})}{f_1(e^{\lambda_1} - e^{\lambda_2}) - \lambda_2 e^{\lambda_1} + \lambda_1 e^{\lambda_2}} \qquad (27)$$

$$\frac{\bar{\theta}_{2o}(s)}{\bar{\theta}_{2i}(s)} = \frac{(\lambda_1 - \lambda_2) e^{\lambda_1 + \lambda_2}}{f_1(e^{\lambda_1} - e^{\lambda_2}) - \lambda_2 e^{\lambda_1} + \lambda_1 e^{\lambda_2}} \qquad (28)$$

where

$$\lambda_1, \lambda_2 = (f_1 - f_2)/2 \pm \sqrt{(f_1 + f_2)^2 - 4 g_1 g_2}/2 \qquad (29)$$

with f_1, f_2, g_1, g_2 the same as given by (24).

If the heat capacities of the tube and shell walls can be neglected, ϕ and ϕ_s do not enter into the picture, and (13) reduces to

$$\frac{\partial \theta_1}{\partial \tau} \pm \frac{\partial \theta_1}{\partial \xi} = a_1' (\theta_2 - \theta_1)$$
$$r\frac{\partial \theta_2}{\partial \tau} + \frac{\partial \theta_2}{\partial \xi} = a_2' (\theta_1 - \theta_2) \qquad (30)$$

where

$$a_1' = U_2 A_2 L/u_1 S_1 \rho_1 c_1, \quad a_2' = U_2 A_2 L/u_2 S_2 \rho_2 c_2 \qquad (31)$$

$$U = \frac{1}{A_2/h_1 A_1 + 1/h_2 + y A_2/k A_m}$$

The transfer functions for this simpler case are the same in form as those given by (19) to (29) with the following modifications for f_1, f_2, g_1, and g_2:

$$f_1 = a_1' + s, \qquad f_2 = a_2' + rs$$
$$g_1 = a_1' \qquad g_2 = a_2' \qquad (32)$$

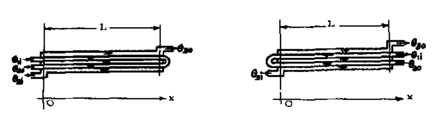

Fig. 2. *a.* A 1-1 heat exchanger with parallel flow. *b.* A 1-1 heat exchanger with counterflow.

TYPE 3. 1-2 MULTIPASS HEAT EXCHANGER, TEMPERATURE OF BOTH STREAMS CONTINUOUSLY VARYING ALONG THE PATH *(11)*

This type of heat exchanger consists of two tube passes in one shell pass. Two orientations regarding the relative directions of flow of the tube and the shell fluids are possible: the flow direction of the shell fluid may be either parallel or opposite to that of the first tube fluid. The former case is known as the *parallel-counterflow*, and the latter case as *counterparallel flow*. These two cases are shown in Figures 3*a* and 3*b*. Several of the transfer functions were obtained in a different form by Masubuchi (*6*) and by Iscol and Altpeter (*5*).

The differential equations for this type of heat exchanger are

$$\frac{\partial \theta_1}{\partial \tau} \pm \frac{\partial \theta_1}{\partial \xi} = a_1(\phi_1 - \theta_1)$$
$$\frac{\partial \theta_2}{\partial \tau} \mp \frac{\partial \theta_2}{\partial \xi} = a_1(\phi_2 - \theta_2)$$
$$r\frac{\partial \theta_3}{\partial \tau} + \frac{\partial \theta_3}{\partial \xi} = a_2(\phi_1 - \theta_3) + a_2(\phi_2 - \theta_3) + a_s(\phi_s - \theta_3)$$
$$\frac{\partial \phi_1}{\partial \tau} = b_1(\theta_1 - \phi_1) + b_2(\theta_3 - \phi_1)$$
$$\frac{\partial \phi_2}{\partial \tau} = b_1(\theta_2 - \phi_2) + b_2(\theta_3 - \phi_2)$$
$$\frac{\partial \phi_s}{\partial \tau} = b_s(\theta_3 - \phi_s) \qquad (33)$$

where

$$a_1 = \frac{h_1 A_1 L}{u_1 S_1 \rho_1 c_1}, a_2 = \frac{h_2 A_2 L}{u_3 S_3 \rho_3 c_3}, a_s = \frac{h_s A_s L}{u_3 S_3 \rho_3 c_3},$$
$$b_1 = \frac{h_1 A_1 L}{u_1 c_t}, b_2 = \frac{h_2 A_2 L}{u_1 c_t}, b_s = \frac{h_s A_s L}{u_1 c_s},$$
$$r = u_1/u_3, \quad \xi = x/L, \quad \tau = t u_1/L \qquad (34)$$

The + sign in the first equation and the − sign in the second equation of (33) apply to the parallel-counterflow case, and the − sign in the first equation and the + sign in the second equation apply to the counterparallel flow case. To reduce the number of parameters to a minimum it has been assumed that the inside heat transfer coefficients for the first and the second tubes are both equal to h_1, and the outside heat transfer coefficients for the two tubes are the same and equal to h_2.

The boundary conditions for the different cases are:

Case (a). Parallel-counterflow, inlet temperature of shell fluid steady, inlet temperature of tube fluid as forcing function:

$$\theta_1(\xi,\tau=0)=\theta_2(\xi,\tau=0)=\theta_3(\xi,\tau=0)=0$$
$$\phi_1(\xi,\tau=0)=\phi_2(\xi,\tau=0)=\phi_s(\xi,\tau=0)=0$$
$$\theta_1(\xi=0,\tau)=\delta(\tau)$$
$$\theta_3(\xi=0,\tau)=0$$
$$\theta_1(\xi=1,\tau)=\theta_2(\xi=1,\tau) \tag{35}$$

Case (b). Parallel-counterflow, inlet temperature of tube fluid steady, inlet temperature of shell fluid as forcing function:

$$\theta_1(\xi,\tau=0)=\theta_2(\xi,\tau=0)=\theta_3(\xi,\tau=0)=0$$
$$\phi_1(\xi,\tau=0)=\phi_2(\xi,\tau=0)=\phi_s(\xi,\tau=0)=0$$
$$\theta_1(\xi,0,\tau)=0$$
$$\theta_3(\xi=0,\tau)=\delta(\tau)$$
$$\theta_1(\xi=1,\tau)=\theta_2(\xi=1,\tau) \tag{36}$$

Case (c). Counterparallel flow, inlet temperature of shell fluid steady, inlet temperature of tube fluid as forcing function:

$$\theta_1(\xi,\tau=0)=\theta_2(\xi,\tau=0)=\theta_3(\xi,\tau=0)=0$$
$$\phi_1(\xi,\tau=0)=\phi_2(\xi,\tau=0)=\phi_s(\xi,\tau=0)=0$$
$$\theta_1(\xi=1,\tau)=\delta(\tau)$$
$$\theta_3(\xi=0,\tau)=0$$
$$\theta_1(\xi=0,\tau)=\theta_2(\xi=0,\tau) \tag{37}$$

Case (d). Counterparallel flow, inlet temperature of tube fluid steady, inlet temperature of shell fluid as forcing function:

$$\theta_1(\xi,\tau=0)=\theta_2(\xi,\tau=0)=\theta_3(\xi,\tau=0)=0$$
$$\phi_1(\xi,\tau=0)=\phi_2(\xi,\tau=0)=\phi_s(\xi,\tau=0)=0$$
$$\theta_1(\xi=1,\tau)=0$$
$$\theta_3(\xi=0,\tau)=\delta(\tau)$$
$$\theta_1(\xi=0,\tau)=\theta_2(\xi=0,\tau) \tag{38}$$

Solving (33) with the boundary conditions (35) to (38) one obtains the following transfer functions.

Parallelcounter flow

Cases (a) and (b):

$$\frac{\bar\theta_{2o}(s)}{\bar\theta_{1i}(s)}=C_1\beta_1+C_2\beta_2+C_3\beta_3 \tag{39}$$

$$\frac{\bar\theta_{3o}(s)}{\bar\theta_{1i}(s)}=C_1e^{\lambda_1}+C_2e^{\lambda_2}+C_3e^{\lambda_3} \tag{40}$$

$$\frac{\bar\theta_{2o}(s)}{\bar\theta_{3i}(s)}=D_1\beta_1+D_2\beta_2+D_3\beta_3 \tag{41}$$

$$\frac{\bar\theta_{3o}(s)}{\bar\theta_{3i}(s)}=D_1e^{\lambda_1}+D_2e^{\lambda_2}+D_3e^{\lambda_3} \tag{42}$$

Counterparallel Flow

Cases (c) and (d):

$$\frac{\bar\theta_{2o}(s)}{\bar\theta_{1i}(s)}=E_1\alpha_1e^{\lambda_1}+E_2\alpha_2e^{\lambda_2}+E_3\alpha_3e^{\lambda_3} \tag{43}$$

$$\frac{\bar\theta_{3o}(s)}{\bar\theta_{1i}(s)}=E_1e^{\lambda_1}+E_2e^{\lambda_2}+E_3e^{\lambda_3} \tag{44}$$

$$\frac{\bar\theta_{2o}(s)}{\bar\theta_{3i}(s)}=F_1\alpha_1e^{\lambda_1}+F_2\alpha_2e^{\lambda_2}+F_3\alpha_3e^{\lambda_3} \tag{45}$$

$$\frac{\bar\theta_{3o}(s)}{\bar\theta_{3i}(s)}=F_1e^{\lambda_1}+F_2e^{\lambda_2}+F_3e^{\lambda_3} \tag{46}$$

In Equations (39) to (46) λ_1, λ_2, λ_3 are the three roots of the following characteristic equation of (33):

$$\lambda^3+f_2\lambda^2-f_1^2\lambda-f_1^2f_2+2f_1g_1g_2=0 \tag{47}$$

where

$$f_1=a_1(b_2+s)/(b_1+b_2+s)+s$$
$$f_2=2a_2(b_1+s)/(b_1+b_2+s)+a_ss/(b_s+s)+rs$$
$$g_1=a_1b_2/(b_1+b_2+s)$$
$$g_2=a_2b_1/(b_1+b_2+s) \tag{48}$$

α_1, α_2, α_3, β_1, β_2, β_3 in (39) to (46) are given by the following:

$$\alpha_1=g_1/(f_1+\lambda_1),\ \alpha_2=g_1/(f_1+\lambda_2),\ \alpha_3=g_1/(f_1+\lambda_3)$$
$$\beta_1=g_1/(f_1-\lambda_1),\ \beta_2=g_1/(f_1-\lambda_2),\ \beta_3=g_1/(f_1-\lambda_3) \tag{49}$$

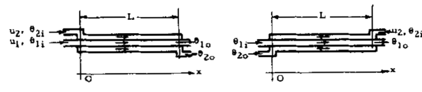

Fig. 3. a. A 1-2 heat exchanger with parallel counterflow. b. A 1-2 heat exchanger with counterparallel flow.

The constants C's, D's, E's, and F's are solved from the following matrix equations:

$$\begin{bmatrix}C_1\\C_2\\C_3\end{bmatrix}=\begin{bmatrix}\alpha_1&\alpha_2&\alpha_3\\(\alpha_1-\beta_1)e^{\lambda_1}&(\alpha_2-\beta_2)e^{\lambda_2}&(\alpha_3-\beta_3)e^{\lambda_3}\\1&1&1\end{bmatrix}^{-1}\begin{bmatrix}1\\0\\0\end{bmatrix} \tag{50}$$

$$\begin{bmatrix}D_1\\D_2\\D_3\end{bmatrix}=\begin{bmatrix}\alpha_1&\alpha_2&\alpha_3\\(\alpha_1-\beta_1)e^{\lambda_1}&(\alpha_2-\beta_2)e^{\lambda_2}&(\alpha_3-\beta_3)e^{\lambda_3}\\1&1&1\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} \tag{51}$$

$$\begin{bmatrix}E_1\\E_2\\E_3\end{bmatrix}=\begin{bmatrix}\beta_1e^{\lambda_1}&\beta_2e^{\lambda_2}&\beta_3e^{\lambda_3}\\(\alpha_1-\beta_1)&(\alpha_2-\beta_2)&(\alpha_3-\beta_3)\\1&1&1\end{bmatrix}^{-1}\begin{bmatrix}1\\0\\0\end{bmatrix} \tag{52}$$

$$\begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix}=\begin{bmatrix}\beta_1e^{\lambda_1}&\beta_2e^{\lambda_2}&\beta_3e^{\lambda_3}\\(\alpha_1-\beta_1)&(\alpha_2-\beta_2)&(\alpha_3-\beta_3)\\1&1&1\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} \tag{53}$$

If the tube and shell wall heat capacities can be neglected, the differential equations for this simpler case take the form

$$\frac{\partial\theta_1}{\partial\tau}\pm\frac{\partial\theta_1}{\partial\xi}=a_1'(\theta_3-\theta_1)$$
$$\frac{\partial\theta_2}{\partial\tau}\mp\frac{\partial\theta_2}{\partial\xi}=a_1'(\theta_3-\theta_2)$$
$$r\frac{\partial\theta_3}{\partial\tau}+\frac{\partial\theta_3}{\partial\xi}=a_2'(\theta_1-\theta_3)+a_2'(\theta_2-\theta_3) \tag{54}$$

where

$$a_1'=U_2A_2L/u_1s_1\rho_1c_1,\quad a_2'=U_2A_2L/u_3s_3\rho_3c_3$$
$$U_2=\frac{1}{A_2/h_1A_1+1/h_2+yA_2/kA_m} \tag{55}$$

The transfer functions for this case are same in form as (39) to (46) with the following different expressions for f_1, f_2, g_1, and g_2:

$$f_1=a_1'+s,\qquad f_2=2a_2'+rs$$
$$g_1=a_1',\qquad g_2=a_2' \tag{56}$$

Numerical computation by hand of the frequency response from the transfer functions of Type 3 exchangers is very time consuming, and machine computation is therefore recommended. The expressions for these transfer functions as listed above are in forms that are readily amenable to machine computation.

The technique used to derive the transfer functions of 1-2 multipass heat exchangers may be extended to analyze an m-n exchanger consisting of m shell passes and n tube passes in each shell pass, but the resulting transfer functions are considerably more complicated.

TYPE 4. ONE FLUID STREAM ISOTHERMAL OR MIXED, THE OTHER UNMIXED

This type of heat exchangers is represented by Figures 4*a* and 4*b*. In Figure 4*a* the stream 2 is an isothermal vapor. Examples of this type of heat exchanger are condensers and steam heaters. In Figure 4*b* the second stream is a thoroughly mixed fluid. The differential equations for these two cases are quite similar.

Several of the transfer functions listed below were obtained by Takahashi (9) and Gould (4). Type 4A heat exchanger has been studied experimentally by DeBolt (3) and by Cohen and Johnson (2) with respect to the transfer function represented by (60).

Type 4 A (Figure 4a)

The differential equations for this type of heat exchanger are

$$\frac{\partial \theta_1}{\partial \tau} + \frac{\partial \theta_1}{\partial \xi} = a_1(\phi - \theta_1)$$

$$\frac{\partial \phi}{\partial \tau} = b_1(\theta_1 - \phi) + b_2(\theta_2 - \phi)$$

$$\frac{\partial \theta_2}{\partial \tau} - \gamma w_2 = a_2 \int_0^1 (\phi - \theta_2)\, d\xi + a_s(\phi_s - \theta_2)$$

$$\frac{d\phi}{\partial \tau} = b_s(\theta_2 - \phi_s)$$

$$w_2 = (\theta_B - \theta_2)/R \tag{57a}$$

where

$$a_1 = h_1A_1L/u_1S_1\rho_1c_1,$$
$$a_2 = h_2A_2L^2/u_1C,$$
$$a_s = h_sA_sL^2/u_1C,$$
$$b_1 = h_1A_1L/u_1c_t,$$
$$b_2 = h_2A_2L/u_1c_t,$$
$$b_s = h_sA_sL/u_1c_s,$$
$$\gamma = L\Lambda/u_1C,$$
$$\xi = x/L,$$
$$\tau = tu_1/L \tag{57b}$$

A few words must be said about C and R, the equivalent heat capacity of the vapor space and the resistance to the vapor flow as defined by the last equation of (57*a*). To obtain a proper expression for C material and energy balances must be applied to the vapor space. The resistance R is determined by the gas flow laws.

The boundary conditions for this heat exchanger follow.

Case (a). Inlet temperature of tube fluid steady and the temperature (or pressure) of the vapor source as forcing function:

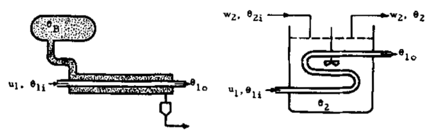

Fig. 4. *a*. Type 4A heat exchanger, one fluid being isothermal vapor and the other unmixed. *b*. Type 4B heat exchanger, one fluid mixed and the other unmixed.

$$\theta_1(\xi, \tau = 0) = \theta_2(\tau = 0) = \theta_B(\tau = 0) = 0$$
$$\phi(\xi, \tau = 0) = \phi_s(\xi, \tau = 0) = 0$$
$$\theta_1(\xi = 0, \tau) = 0$$
$$\theta_B(\tau) = \delta(\tau) \tag{58}$$

Case (b). Temperature (or pressure) of the vapor source steady, and inlet temperature of the tube fluid as forcing function:

$$\theta_1(\xi, \tau = 0) = \theta_2(\tau = 0) = \theta_B(\tau = 0) = 0$$
$$\phi(\xi, \tau = 0) = \phi_s(\xi, \tau = 0) = 0$$
$$\theta_1(\xi = 0, \tau) = \delta(\tau)$$
$$\theta_B(\tau) = 0 \tag{59}$$

Solving (57*a*) with the boundary conditions (58) and (59) one obtains the following transfer functions.

Case (a):

$$\frac{\bar{\theta}_{1o}(s)}{\bar{\theta}_2(s)} = \frac{g_1}{f_1}(1 - e^{-f_1}) \tag{60}$$

$$\frac{\bar{\theta}_2(s)}{\bar{\theta}_B(s)} = \frac{\gamma f_1^2}{R[f_1(f_1f_2 - g_1g_2) + g_1g_2(1 - e^{-f_1})]} \tag{61}$$

$$\frac{\bar{\theta}_{1o}(s)}{\bar{\theta}_B(s)} = \frac{\gamma f_1 g_1(1 - e^{-f_1})}{R[f_1(f_1f_2 - g_1g_2) + g_1g_2(1 - e^{-f_1})]} \tag{62}$$

Case (b):

$$\frac{\bar{\theta}_2(s)}{\bar{\theta}_{1i}(s)} = \frac{f_1g_2(1 - e^{-f_1})}{f_1(f_1f_2 - g_1g_2) + g_1g_2(1 - e^{-f_1})} \tag{63}$$

$$\frac{\bar{\theta}_{1o}(s)}{\bar{\theta}_{1i}(s)} = e^{-f_1} + \frac{g_1g_2(1 - e^{-f_1})^2}{f_1(f_1f_2 - g_1g_2) + g_1g_2(1 - e^{-f_1})} \tag{64}$$

where

$$f_1 = a_1(b_2 + s)/(b_1 + b_2 + s) + s$$
$$f_2 = \gamma/R + a_2(b_1 + s)/(b_1 + b_2 + s) + a_ss/(b_s + s) + s$$
$$g_1 = a_1b_2/(b_1 + b_2 + s)$$
$$b_2 = a_2b_1/(b_1 + b_2 + s) \tag{65}$$

If the vapor space is assumed to be at saturated conditions, the pressure in the shell space is, within a narrow range, related linearly to the vapor temperature. Under such an assumption (61) and (62), modified by a multiplying constant, may be regarded as the transfer functions relating the pressure of the vapor source to the pressure in the shell space of the heat exchanger and to the outlet temperature of the tube fluid.

If the heat capacities of the solid walls can be neglected, the same transfer functions are obtained with the following modifications for f_1, f_2, g_1, and g_2:

$$f_1 = a_1' + s, \quad f_2 = a_2' + \gamma/R + s$$
$$g_1 = a_1', \qquad g_2 = a_2'$$
$$a_1' = U_2A_2L/u_1S_1\rho_1c_1,$$
$$a_2' = U_2A_2L^2/u_1C \tag{66}$$

Type 4B (Figure 4b)

The differential equations for this type of heat exchanger are

$$\frac{\partial \theta_1}{\partial \tau} + \frac{\partial \theta_1}{\partial \xi} = a_1(\phi - \theta_1)$$

$$\frac{\partial \phi}{\partial \tau} = b_1(\theta_1 - \phi) + b_2(\theta_2 - \phi)$$

$$r\frac{d\theta_2}{\partial \tau} + \theta_2 - \theta_{2i} = a_2 \int_0^1 (\phi - \theta_2) d\xi + a_s(\phi_s - \theta_2)$$

$$\frac{d\phi_s}{d\tau} = b_s(\theta_2 - \phi_s), \tag{67}$$

where

$$a_1 = h_1A_1L/u_1S_1\rho_1c_1,$$
$$a_2 = h_2A_2/w_2c_2,$$
$$a_s = h_sA_s/w_2c_2$$
$$b_1 = h_1A_1L/u_1c_t,$$
$$b_2 = h_2A_2L/u_1c_t,$$
$$b_s = h_sA_s/u_1C_s$$
$$r = M_2u_1/w_2L,$$
$$\xi = x/L,$$
$$\tau = tu_1/L \tag{68}$$

The similarities between the differential equations of Type 4A and Type 4B exchangers are quite obvious from (57*a*) and (67). The transfer functions for this type of exchanger are of the same form as (61) to (64) with the following modifications:

$$f_2 = 1 + a_2(b_1 + s)/(b_1 + b_2 + s) + a_ss/(b_s + s) + rs \tag{69}$$

$\frac{\bar{\theta}_2(s)}{\bar{\theta}_{2i}(s)}$ same as (61) without γ and R

$\frac{\bar{\theta}_{1o}(s)}{\bar{\theta}_{2i}(s)}$ same as (62) without γ and R

$\frac{\bar{\theta}_2(s)}{\bar{\theta}_{1i}(s)}$ same as (63)

$\frac{\bar{\theta}_{1o}(s)}{\bar{\theta}_{1i}(s)}$ same as (64)

CONCLUSIONS

The transfer functions given above were obtained by straightforward and rigorous procedures of mathematical analysis. They are valid under the assumptions underlying the differential equations describing the system. Experimental dynamic tests conducted on various types of heat exchangers (*2, 5, 6, 9, 11*) seem to support the validity of the theoretical transfer functions presented here.

Most of the transfer functions of the heat exchangers are complicated in form, and the numerical computation of the frequency response from these transfer functions would be very tedious if done by hand. Fortunately the voluminous computational work can be accomplished quickly by a modern high-speed computer. Therefore, with the aid of a computer, these transfer functions, though complicated in form, are practical for predicting the dynamic behavior of the system.

It has been suggested that some simple empirical expressions of transfer functions may be developed to approximate the dynamics of the heat exchanger. However this must be done with a true insight in the dynamics of the system obtained through rigorous analysis of the system. Some efforts (*7, 10*) have been made in this direction. It is hoped that the transfer functions presented here may be used as a guide or comparison standard for the simpler expressions proposed.

NOTATION

A = surface area of heat transfer, sq. ft.; sometimes the lateral area of a unit length of tube for heat transfer, sq. ft./ft.
a = dimensionless parameter; for the specific expression for a given case, see text
b = dimensionless parameter, see text
C = equivalent heat capacity for the whole space occupied by fluid stream 2, B.t.u./°F.
C_1, C_2, C_3 = integration constants, see text
C_s = heat capacity for the whole shell wall, B.t.u./°F.
c_s = heat capacity for the shell wall per unit length, B.t.u./°F., ft.
c_t = heat capacity for the tube wall per unit length, B.t.u./°F., ft.
c_w = specific heat of partition wall, B.t.u./°F., lb.
c = specific heat of fluid, B.t.u./°F., lb.
D_1, D_2, D_3 = integration constants, see text
E_1, E_2, E_3 = integration constants, see text
e = Napierian base
F_1, F_2, F_3 = integration constants, see text
f_1, f_2 = intermediate parameters, dimensionless, see text
g_1, g_2 = intermediate parameters, dimensionless, see text
h = heat transfer coefficient, B.t.u./hr., sq. ft., °F.
j = square root of -1
k = heat conductivity, B.t.u./hr., ft., °F.
L = length of the heat exchanger, ft.
M = fluid holdup of one portion of the heat exchanger, lb.
M_w = mass of partition wall, lb.
R = resistance to flow defined by the last equation of (41).
r = ratio of fluid stream velocities, dimensionless
S = cross-sectional area of tube or shell available for fluid flow, sq. ft.
s = LaPlace transform variable with respect to τ
T = specially defined period of time, hr. see text
t = time, hr.
U = overall heat transfer coefficient, B.t.u./(hr.-sq. ft.-°F.)
u = fluid velocity of flow, ft./hr.
w = mass rate of flow of fluid, lb./hr.
x = distance along the exchanger, usually taken as positive in the direction of flow of the shell fluid, ft.
y = distance along the direction of flow of a second fluid perpendicular to that of the first fluid; sometimes the thickness of the tube wall, ft.

Greek Letters

$\alpha_1, \alpha_2, \alpha_3$ = integration constants, see Equation (49)
$\beta_1, \beta_2, \beta_3$ = integration constants, see Equation (49)
γ = defined by Equation (57*b*)
δ = unit impulse function
θ = bulk temperature of the fluid, a function of t or τ only, if the temperature of the fluid does not change along the length of the exchanger; a function of x and t or ξ and τ, if the temperature of the fluid changes along the length of exchanger, °F.
ϕ = temperature of the tube or shell wall, see above note, °F.
$\bar{\theta}$ = Laplace transform of θ, sometimes written as $\bar{\theta}(s)$ to indicate that it is a function of s
$\bar{\phi}$ = Laplace transform of ϕ, sometimes written as $\bar{\phi}(s)$ to indicate that it is a function of s
Λ = latent heat of vapor, B.t.u./lb.
λ = root or eigenvalue of the characteristic equation
ρ = density of the fluid, lb./cu. ft.
ξ = dimensionless space variable, $= x/L$
τ = dimensionless time variable, $= t/T$
ω = dimensionless frequency of the sinusoidal exciting function, $\omega = 2\pi fT$, where f is the frequency of the exciting function in cycles per unit time, and T is the arbitrary defined time period which converts the actual t to τ by the relation $\tau = t/T$

Subscripts

1,2,3 = the first, second, third stream of fluid when used with θ, ϕ, c, u, w, ρ, S; the inside and outside surface area of the tube when used with A and h; sometimes different dimensionless parameters and integration constants
B = boiler or vapor source
i = inlet condition
m = log mean or arithmetic mean
o = outlet condition
s = shell surface area in contact with the shell fluid, sometimes the whole shell wall
w = partition wall

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